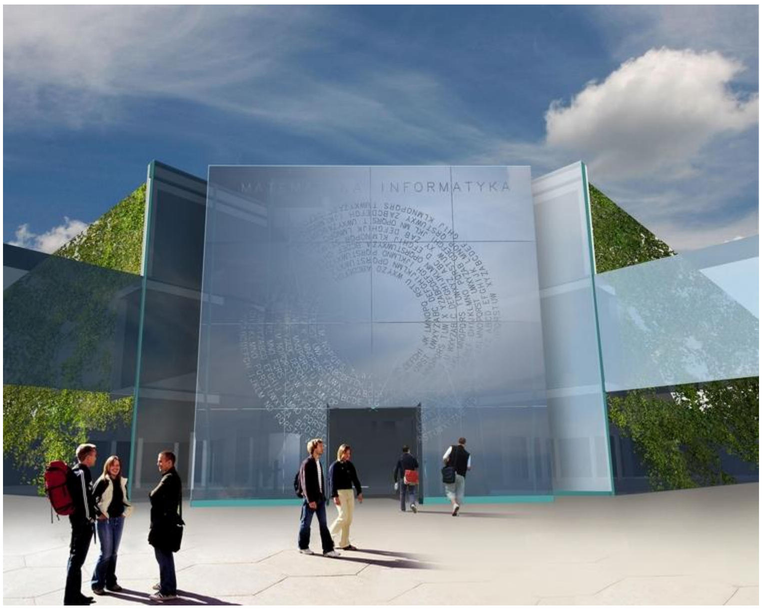
Ore localizations of nearrings

Małgorzata E. Hryniewicka

Institute of Mathematics, University of Białystok
Ciołkowskiego 1M, 15-245 Białystok, Poland
e-mail: margitt@math.uwb.edu.pl



Example. Let G be a (not necessarily abelian) group written additively, and let

$$M_o(G) = \{f\colon G \to G \mid f(0) = 0\}.$$

For any $f, g \in M_0(G)$, let $f + g\colon G \to G$ be defined by

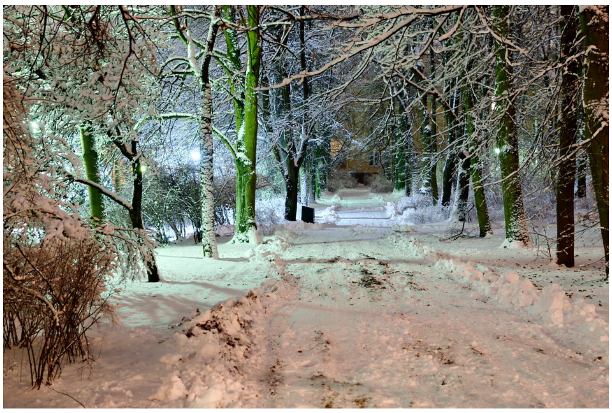
$$(f + g)(x) = f(x) + g(x)$$

for every $x \in G$. The set $M_0(G)$ is a group. The additional assumption on the abelianity of the group G implies the abelianity of the group $M_0(G)$.

For any $f, g \in M_0(G)$, let $f \circ g\colon G \to G$ be defined by

$$(f \circ g)(x) = f(g(x))$$

for every $x \in G$. The set $M_0(G)$ is a semigroup with unit.

Białystok, the Planty Park in winter

The right distributive condition is always fulfilled: for any $f, g, h \in M_0(G)$ and $x \in G$,

$$
\begin{aligned}
((f+g)\circ h)(x) &= (f\circ h)(x) + (g\circ h)(x) \\
&= (f\circ h + g\circ h)(x),
\end{aligned}
$$

while the left distributive condition

$$
\begin{aligned}
(f\circ(g+h))(x) &= f(g(x)+h(x)) \\
&\ ?\ (f\circ g)(x) + (f\circ h)(x) \\
&= (f\circ g + f\circ h)(x)
\end{aligned}
$$

may not be fulfilled as f is not a group endomorphism.

This algebraic structure $(M_0(G), +, \circ)$ is a classic example of a nearring (more precisely, a zerosymmetric right nearring with unit). $\square$

Białystok, the Planty Park in winter

Example. Let R be a ring, let V be a left module over R, and let

$$M_{hom}(V) = \{f\colon V \to V \mid f(rx) = rf(x) \text{ for all } r \in R \text{ and } x \in V\}.$$

The set $M_{hom}(V)$ together with the pointwise addition and map composition forms a nearring. $\square$


Białystok, The Branicki Palace

·

Definition. By *a nearring* we mean a set N of no fewer than two elements together with two binary operations, the addition and multiplication, in which

(1) N is a (not necessarily abelian) group with respect to the addition.

(2) N is a semigroup with unit with respect to the multiplication.

(3) $(k+m)n = kn + mn$ for all $k, m, n \in N$.

(4) $n0 = 0n = 0$ for every $n \in N$.

We say that a nearring is *abelian* (respectively, *commutative*) if the additive group mentioned above is abelian (respectively, the multiplicative semigroup mentioned above is commutative). $\square$

Definition (James A. Graves, Joseph J. Malone, 1973). Let N be a nearring, and let $S \subseteq N$ be a multiplicatively closed set. By *a nearring of right quotients of N with respect to S* we mean a nearring N_S together with an embedding $\phi\colon N \hookrightarrow N_S$ for which

(1) $\phi(s)$ is invertible in N_S for every $s \in S$.

(2) every element of N_S is of the form $\phi(n)\phi(s)^{-1}$ where $n \in N$ and $s \in S$. $\square$

Białystok, The Branicki Palace

Theorem (James A. Graves, Joseph J. Malone, 1973). Let N be a nearring, and let $S \subseteq N$ be a multiplicatively closed set of both left and right cancellable elements in N. Then a nearring of right quotients N_S exists iff N satisfies *the right Ore condition with respect to* S

$$sn_1 = ns_1 \text{ for all } n \in N,\ s \in S, \\ \text{and for some } n_1 \in N,\ s_1 \in S.$$

□

If S is the set of all both left and right cancellable elements in N, then the nearring N_S is called *the total nearring of right quotients of* N.

The Podlasie Province

Theorem (James A. Graves, Joseph J. Malone, 1973). If a nearring N satisfies the left cancellation condition then

(1) N has no proper zero divisors.

(2) N satisfies the right cancellation condition. $\square$

Definition (James A. Graves, Joseph J. Malone, 1973). By *a near domain* we mean a nearring N satisfies both the left cancellation condition and the right Ore condition. $\square$

Theorem (James A. Graves, Joseph J. Malone, 1973). Every near domain N has the total nearring of right quotients N_S where $S = N \setminus \{0\}$. $\square$

The Podlasie Province

The left analogue of the notion of a nearring of right quotients in the sense of Graves and Malone is defined similarly. Unfortunately, as pointed out by Carlton J. Maxson, the Ore construction does not hold for a nearring of left quotients in the sense of Graves and Malone ${}_SN$, because a substitute for the left distributive condition in N is necessary for the addition in ${}_SN$ to be well defined.

As we said, the right distributive condition being found in the definition of a nearring has not got its left equivalent. An effect of this is the asymmetry of the conditions guaranteeing the existence of nearrings of left and of right quotients respectively. For this reason and for the better comprehension of every distributive condition, we shall require neither distributive condition.


The Podlasie Province

.

An early morning in the Białowieża National Park

Definition. By *a nondistributive ring* we shall mean a set N of no fewer than two elements together with two binary operations, the addition and multiplication, in which

(1) N is a (not necessarily abelian) group with respect to the addition.

(2) N is a semigroup with unit with respect to the multiplication.

(3) $n0 = 0n = 0$ for every $n \in N$.

We shall say that a nondistributive ring is *abelian* (respectively, *commutative*) if the additive group mentioned above is abelian (respectively, the multiplicative semigroup mentioned above is commutative).

$\square$

Definition. Let N be a nondistributive ring, and let $S \subseteq N$ be a multiplicatively closed set. We shall call a nondistributive ring $S^{-1}N$ *a nondistributive ring of left quotients of N with respect to S* if there exists a homomorphism $\eta\colon N \to S^{-1}N$ of nondistributive rings for which

(1) $\eta(s)$ is invertible in $S^{-1}N$ for every $s \in S$.

(2) $\eta(s)$ is left distributive in $S^{-1}N$ for every $s \in S$.

(3) every element of $S^{-1}N$ is of the form $\eta(s)^{-1}\eta(n)$ where $n \in N$ and $s \in S$.

(4) $\ker \eta = \left\{n \in N \mid r(s+n) = rs \text{ for some } r, s \in S\right\}$. $\square$

Augustów

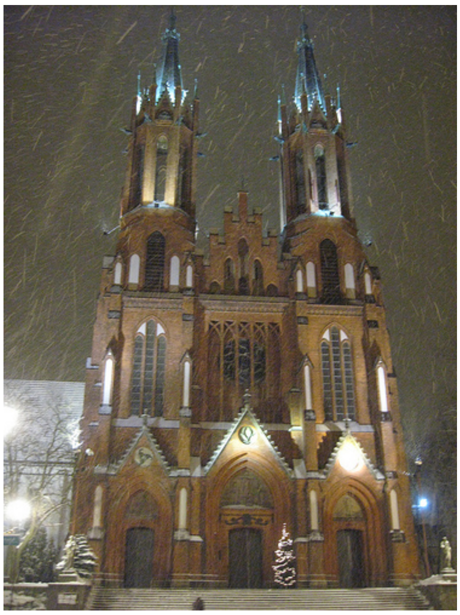
For a multiplicatively closed set S in a nondistributive ring N, we let

$$U = \{n \in N \mid r_2(s_2 + nr_1 - s_1) = r_2s_2 \\ \text{for some } r_2, s_2 \in S\} \supseteq S.$$

Theorem. A nondistributive ring N has a nondistributive ring of left quotients $S^{-1}N$ with respect to a multiplicatively closed set $S \subseteq N$ iff S satisfies the following conditions

(1) for all $n \in N$ and $s \in S$ there exist $n_1 \in N$ and $s_1, r_2, s_2 \in S$ such that $r_2(s_2+n_1s-s_1n) = r_2s_2$.

(2) for all $m, n \in N$ and $s \in U$ there exist $r_1, s_1 \in S$ such that $r_1(s_1 + s(m + n) - sn - sm) = r_1s_1$.

Białystok, the Farny Church

(3) for all $m, n \in N$ if $r(s + tmu - tnu) = rs$ for some $r, s, t, u \in S$, then $r_1(s_1 + m - n) = r_1 s_1$ for some $r_1, s_1 \in S$.

(4) for all $m, n \in N$ if $r(s+m) = rs$ and $t(u+n) = tu$ for some $r, s, t, u \in S$, then $r_1(s_1+m-n) = r_1 s_1$ for some $r_1, s_1 \in S$.

(5) for all $m, n \in N$ if $r(s + n) = rs$ for some $r, s \in S$, then $r_1(s_1 + m + n - m) = r_1 s_1$ for some $r_1, s_1 \in S$.

(6) for all $k, l, m, n \in N$ if $r(s+m-n) = rs$ for some $r, s \in S$, then $r_1(s_1+kml-knl) = r_1 s_1$ for some $r_1, s_1 \in S$.

The additional assumption that N is an abelian nondistributive ring (respectively, a commutative nondistributive ring, a left nearring, a right nearring) implies the same for $S^{-1}N$. $\square$

For a multiplicatively closed set S in a nondistributive ring N, we shall call a homomorphism $\eta\colon N \to M$ of nondistributive rings *S-inverting* (respectively, *S-left distributing*) if $\eta(s)$ is invertible in M (respectively, left distributive in M) for every $s \in S$.

Theorem. Under conditions (1)–(6) stated above, $\eta\colon N \to S^{-1}N$ is a universal both S-inverting and S-left distributing homomorphism of nondistributive rings. $\square$

We shall symmetrically define the notion of a nondistributive ring of right quotients.

Theorem. If a nondistributive ring N has nondistributive rings of left and of right quotients respectively, with respect to a multiplicatively closed set $S \subseteq N$, then $S^{-1}N \cong NS^{-1}$. $\square$


Białystok in winter

.

Białystok, Kościuszko Square

An example of a nearing of left quotients $\boldsymbol{S^{-1}N}$ of a nearing $\boldsymbol{N}$ with respect to a multiplicatively closed set $\boldsymbol{S} \subseteq \boldsymbol{N}$ with a non-injective homomorphism $\eta \colon \boldsymbol{N} \to \boldsymbol{S^{-1}N}$ of nearrings. For a fixed odd prime number p, let $R = \mathbb{Z}/2p\mathbb{Z}$ be a quotient ring, and let $S = U(R) \cup \{\overline{p}\} = \{\overline{1}, \overline{3}, \ldots, \overline{2p-1}\}$ be a multiplicatively closed set in R. On $S \times R$ we define an equivalence relation as follows

$$\begin{aligned}&(\overline{s}, \overline{n}) \sim (\overline{s'}, \overline{n'}) \text{ in } S \times R \text{ iff}\\ &\overline{r} \cdot \overline{s} = \overline{r'} \cdot \overline{s'} \text{ and } \overline{r} \cdot \overline{n} = \overline{r'} \cdot \overline{n'} \text{ for some } \overline{r}, \overline{r'} \in S.\end{aligned}$$

By a left quotient $\overline{s} \backslash \overline{n}$ we mean the equivalence class containing $(\overline{s}, \overline{n}) \in S \times R$, and by $S^{-1}R$ the set of all equivalence classes under the relation $\sim$. In $S^{-1}R$ we consider

$$\begin{aligned} S^{-1}S &= \{\overline{s}\backslash\overline{n} \in S^{-1}R \mid \overline{s}\backslash\overline{n} = \overline{r}\backslash\overline{t} \text{ for some } \overline{r}, \overline{t} \in S\} \\ &= \{\overline{s}\backslash\overline{n} \in S^{-1}R \mid \overline{r} \cdot \overline{n} \in S \text{ for some } \overline{r} \in S\}. \end{aligned}$$

For any $\overline{r}\backslash\overline{k} \in S^{-1}S$ and $\overline{s}\backslash\overline{m}, \overline{t}\backslash\overline{n} \in S^{-1}R$ we define the addition of quotients from $S^{-1}R$ and the left multiplication of quotients from $S^{-1}R$ by quotients from $S^{-1}S$ as follows

$$\overline{s}\backslash\overline{m} + \overline{t}\backslash\overline{n} = \overline{t_1} \cdot \overline{s}\backslash(\overline{t_1} \cdot \overline{m} + \overline{s_1} \cdot \overline{n})$$

where $\overline{s_1} \cdot \overline{t} = \overline{t_1} \cdot \overline{s}$ for some $\overline{s_1}, \overline{t_1} \in S$ and

$$\overline{r}\backslash\overline{k} \cdot \overline{t}\backslash\overline{n} = \overline{t_2} \cdot \overline{u} \cdot \overline{r}\backslash\overline{k_2} \cdot \overline{n}$$

where $\overline{u} \cdot \overline{k} \in S$ and $\overline{t_2} \cdot \overline{u} \cdot \overline{k} = \overline{k_2} \cdot \overline{t}$ for some $\overline{k_2}, \overline{t_2} \in S$.

Both the definitions are independent of the choice of the representations for the equivalence classes, which enables us to assume without restriction of generality that $\overline{k} \in S$. The set $S^{-1}R$ is an abelian additive group, the set $S^{-1}S$ is an abelian multiplicatively group, and the group $S^{-1}S$ acts faithfully on $S^{-1}R$ as group automorphisms according to the rule

$$\overline{r}\backslash\overline{s} \rightharpoonup \overline{t}\backslash\overline{n} = \overline{r}\backslash\overline{s} \cdot \overline{t}\backslash\overline{n}$$

for all $\overline{n} \in R$ and $\overline{r}, \overline{s}, \overline{t} \in S$. We consider

$$\begin{aligned}\boldsymbol{N} = \{f \in M_0(R) \mid \overline{p} \cdot \overline{s} \cdot f(\overline{n}) = \overline{p} \cdot f(\overline{s} \cdot \overline{n}) \\ \text{for all } \overline{n} \in R \text{ and } \overline{s} \in S\} \\ = \{f \in M_0(R) \mid f(R \setminus S) \subseteq R \setminus S \text{ and either} \\ f(S) \subseteq R \setminus S \text{ or } f(S) \subseteq S\}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{R} = \{f \in M_0(R) \mid f(\overline{n}) = \overline{0} \text{ for every } \overline{n} \in R \setminus S \\ \text{and } f(\overline{s}) = \overline{r} \cdot \overline{s} \text{ for some } \overline{r} \in S \\ \text{and every } \overline{s} \in S\} \subseteq \boldsymbol{N}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{Q} = \{q \in M_0(S^{-1}R) \mid \overline{r}\backslash\overline{s} \cdot q(\overline{t}\backslash\overline{n}) = q(\overline{r}\backslash\overline{s} \cdot \overline{t}\backslash\overline{n}) \\ \text{for all } \overline{n} \in R \text{ and } \overline{r}, \overline{s}, \overline{t} \in S\}.\end{aligned}$$

The set $\boldsymbol{Q}$ is a nearing of left quotients of the nearing $\boldsymbol{N}$ with respect to the multiplicatively closed set $\boldsymbol{S} = \boldsymbol{R} \cup \{id_R\} \subseteq \boldsymbol{N}$ with some homomorphism $\eta\colon \boldsymbol{N} \to \boldsymbol{Q}$ of nearrings for which

$$\ker \eta = \{f \in M_0(R) \mid f(R) \subseteq R \setminus S\}.$$

$\square$

A sunrise in the Białowieża National Park